

# ON FINITE-SHEETED COVERING MAPPINGS ONTO SOLENOIDS

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**ABSTRACT.** We study limit mappings from a solenoid onto itself. It is shown that each equivalence class of finite-sheeted covering mappings from connected topological spaces onto a solenoid is determined by a limit mapping. Properties of periodic points of limit mappings are also studied.

## INTRODUCTION

There are various ways one can view solenoids (see, e.g., [4], [11, (10.12)]). A solenoid may be defined as follows. Let  $P = (p_1, p_2, \dots)$  be a sequence of prime numbers (1 not being included as a prime). The  $P$ -adic solenoid  $\Sigma_P$  is the inverse limit of the inverse sequence

$$\mathbb{S}^1 \xleftarrow{f_1^2} \mathbb{S}^1 \xleftarrow{f_2^3} \mathbb{S}^1 \xleftarrow{f_3^4} \dots, \quad (1)$$

where  $\mathbb{S}^1$  is the unit circle considered as a subspace of the space  $\mathbb{C}$  of all complex numbers endowed with the natural topology, and every bonding mapping  $f_n^{n+1}$  is given by  $f_n^{n+1}(z) = z^{p_n}$  for each  $z \in \mathbb{S}^1$ ,  $n \in \mathbb{N}$ . If  $p_n = 2$  for all  $n \in \mathbb{N}$ , then the solenoid  $\Sigma_P$  is said to be *dyadic*. As is well known, the  $P$ -adic solenoid is a metric continuum which is not locally connected at any point. Sequences  $P$  and  $Q$  of prime numbers are said to be *equivalent* (written  $P \sim Q$ ) if a finite number of terms can be deleted from each sequence so that every prime number occurs the same number of times in the deleted sequences. The solenoids  $\Sigma_P$  and  $\Sigma_Q$  are homeomorphic iff  $P \sim Q$  [2], [14] (see also [1]). The solenoid is a compact abelian group under the coordinatewise multiplication with the identity  $(1, 1, \dots)$ . The condition  $P \sim Q$  is also a criterion for existing of topological isomorphism between the topological groups  $\Sigma_P$  and  $\Sigma_Q$  [2].

For given  $k \in \mathbb{N}$  let us consider the limit mapping  $h_P^k : \Sigma_P \rightarrow \Sigma_P$  induced by the mapping  $\{h_n^k : n \in \mathbb{N}\}$  between two copies of (1) :

$$\begin{array}{ccccccc} \mathbb{S}^1 & \xleftarrow{f_1^2} & \mathbb{S}^1 & \xleftarrow{f_2^3} & \mathbb{S}^1 & \xleftarrow{f_3^4} & \dots & \Sigma_P \\ h_1^k \downarrow & & \downarrow h_2^k & & \downarrow h_3^k & & \downarrow h_P^k & \\ \mathbb{S}^1 & \xleftarrow{f_1^2} & \mathbb{S}^1 & \xleftarrow{f_2^3} & \mathbb{S}^1 & \xleftarrow{f_3^4} & \dots & \Sigma_P, \end{array} \quad (2)$$

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where  $h_n^k$  is the  $k$ -th potency mapping for each  $n \in \mathbb{N}$ , that is,  $h_n^k(z) = z^k$  for every  $z \in \mathbb{S}^1$ .

This note deals with the limit mappings  $h_P^k$ . Note that, for each  $k \in \mathbb{N}$ , the limit mapping  $h_{(2,2,\dots)}^k$  of the dyadic solenoid is a covering mapping [17], [3]. Theorem 19 of [3] states that the dyadic solenoid admits any odd degree  $k$  covering mapping of the form  $h_{(2,2,\dots)}^k$ , while it does not admit any covering mapping  $h_{(2,2,\dots)}^k$  of an even degree  $k$ . For each  $k \in \mathbb{N}$  the set of periodic points of  $h_{(2,2,\dots)}^k$  is dense in  $\Sigma_{(2,2,\dots)}$  [17], [3].

The purpose of this note is to extend the above results concerning  $h_{(2,2,\dots)}^k$  to the limit mappings  $h_P^k$  of an arbitrary  $P$ -adic solenoid.

## 1. PRELIMINARIES

In this section we establish some notation that is used throughout. As usual, we denote by  $\mathbb{N}$  the set of all positive integers. For a complex number  $z \in \mathbb{C}$  and  $n \in \mathbb{N}$  we denote by  $\sqrt[n]{z}$  the set of all values of the  $n$ -th root of  $z$ .

It is well known that in studying of finite-sheeted covering mappings from connected topological spaces onto solenoids there is no loss of generality in assuming that covering spaces are metric continua, i.e., metric compact connected spaces. So all topological spaces are assumed to be metric. A *mapping* between two spaces always means a continuous function. For basic notions of the theory of inverse limit spaces we refer the reader to [5, Chapter VIII] and [6, Chapter 2]. Let  $\{X_n, \pi_n^{n+1}\}$  and  $\{Y_n, \rho_n^{n+1}\}$  be inverse sequences of spaces  $X_n$  and  $Y_n$  with bonding mappings  $\pi_n^{n+1} : X_{n+1} \rightarrow X_n$  and  $\rho_n^{n+1} : Y_{n+1} \rightarrow Y_n$ , where  $n \in \mathbb{N}$ . We denote by  $X_\infty$  and  $Y_\infty$  the inverse limit spaces of  $\{X_n, \pi_n^{n+1}\}$  and  $\{Y_n, \rho_n^{n+1}\}$  respectively. Recall that a sequence  $\{\sigma_n : X_n \rightarrow Y_n \mid n \in \mathbb{N}\}$  of mappings is called a *mapping* between the inverse sequences  $\{X_n, \pi_n^{n+1}\}$  and  $\{Y_n, \rho_n^{n+1}\}$  if  $\sigma_n \circ \pi_n^{n+1} = \rho_n^{n+1} \circ \sigma_{n+1}$  for all  $n \in \mathbb{N}$ . Then there exists a mapping  $\sigma_\infty : X_\infty \rightarrow Y_\infty : (x_1, x_2, \dots) \mapsto (\sigma_1(x_1), \sigma_2(x_2), \dots)$ . It is called the *limit mapping* induced by  $\{\sigma_n : n \in \mathbb{N}\}$ .

For any  $m, n \in \mathbb{N}$  satisfying  $m \leq n$ , we denote by  $f_m^n$  the bonding mapping of the inverse sequence (1). Thus, for each  $n \in \mathbb{N}$ , the mapping  $f_n^n$  is the identity on  $\mathbb{S}^1$  and  $f_l^n = f_l^m \circ f_m^n$  for all  $l, m, n \in \mathbb{N}$  satisfying  $l \leq m \leq n$ .

The identity  $(1, 1, \dots)$  of a solenoid is denoted by  $e$ .

Recall that a surjective mapping  $f : X \rightarrow Y$  between spaces  $X$  and  $Y$  is called:

- *$k$ -to-1*, where  $k \in \mathbb{N}$ , provided that  $\text{card } f^{-1}(y) = k$  for each  $y \in Y$ ;

- a *finite-sheeted covering mapping* if it is a  *$k$ -sheeted ( $k$ -fold) covering mapping* for some  $k \in \mathbb{N}$ ; that is, every point  $y \in Y$  has an open neighborhood  $W$  in  $Y$  such that the inverse image  $f^{-1}(W)$  can be written as the union of  $k$  disjoint open subsets of  $X$  each of which is mapped homeomorphically onto  $W$  under  $f$ .

The number  $k$  in the above definitions is called a *degree* of the mapping. Finite-sheeted covering mappings  $f_1 : X_1 \rightarrow Y$  and  $f_2 : X_2 \rightarrow Y$  are said to be *isomorphic* (or *equivalent*) if there is a homeomorphism  $g : X_1 \rightarrow X_2$  such that  $f_1 = f_2 \circ g$ .

Throughout this note  $P = (p_1, p_2, \dots)$  denotes a sequence of prime numbers. We say that a prime number  $p$  occurs *infinitely often* in  $P$  if  $p = p_n$  for infinitely many terms  $p_n$ ,  $n \in \mathbb{N}$ . We denote by  $S(P)$  a subset of  $\mathbb{N} \setminus \{1\}$  consisting of all prime numbers which do not occur infinitely often in  $P$ . In other words,  $q \in S(P)$  iff  $q > 1$  is a prime number such that, for some  $m \in \mathbb{N}$ , we have  $q \neq p_n$  for all  $n \geq m$ .

## 2. COVERING MAPPINGS.

The following proposition is an analog of [17, Lemma 1] and [3, Proposition 8].

**Proposition 1.** *For each  $k \in \mathbb{N}$  the limit mapping  $h_P^k : \Sigma_P \rightarrow \Sigma_P$  is a finite-sheeted covering mapping, and its degree is at most  $k$ .*

*Proof.* Let us fix  $k \in \mathbb{N}$ . Clearly, it suffices to show that  $h_P^k$  is an  $m$ -to-1 surjective open mapping for some  $m \leq k$  (cf. [16, Chapter X, §6], [3, Proposition 1]).

Since  $\mathbb{S}^1$  is compact and each mapping  $h_n^k$  in (2) is surjective, the limit mapping  $h_P^k$  is surjective too [6, Theorem 3.2.14]. Because  $h_P^k$  is a continuous homomorphism from a compact group onto itself,  $h_P^k$  is an open mapping [11, (5.29)] and the equality  $\text{card}(h_P^k)^{-1}(y) = \text{card}(h_P^k)^{-1}(e)$  holds for each  $y \in \Sigma_P$ . Note, if  $(x_1, x_2, \dots) \in (h_P^k)^{-1}(e)$ , then  $x_n \in \sqrt[k]{1}$  for all  $n \in \mathbb{N}$ . Using these observations, one can easily see that  $h_P^k$  is an  $m$ -to-1 mapping and  $m \leq k$ .  $\square$

We shall determine degrees of the limit mappings  $h_P^k$ . In order to do this we find out cardinalities of fibers  $(h_P^k)^{-1}(e)$ ,  $k \in \mathbb{N}$ . (See also [3, Statements 15, 17, 18]).

**Proposition 2.** *If  $k$  is a prime number which occurs infinitely often in the sequence  $P$ , then the limit mapping  $h_P^k : \Sigma_P \rightarrow \Sigma_P$  is a homeomorphism.*

*Proof.* We claim that the set  $(h_P^k)^{-1}(e)$  consists of only one point  $e$ . To show this we suppose that  $z = (z_1, z_2, \dots) \in (h_P^k)^{-1}(e)$ . Then  $z_n^k = 1$  for all  $n \in \mathbb{N}$ . By assumption, for given  $n \in \mathbb{N}$ , there is an integer  $m \geq n$  such that  $f_m^{m+1}$  in (1) is the  $k$ -th potency mapping. Therefore, we have

$$z_n = f_n^m(f_m^{m+1}(z_{m+1})) = f_n^m(z_{m+1}^k) = f_n^m(1) = 1.$$

Since these equalities hold for each  $n \in \mathbb{N}$ , it follows that  $z = e$ , as claimed.

Thus  $\text{card}(h_P^k)^{-1}(e) = 1$  and, by Proposition 1, the limit mapping  $h_P^k$  is a homeomorphism.  $\square$

The verification of the following lemma is straightforward (cf. [3, Fact 16]).

**Lemma.** *If  $k = l \cdot m$  for some  $l, m \in \mathbb{N}$ , then  $h_P^k = h_P^l \circ h_P^m$ .*

Combining Lemma and Proposition 2, we have :

**Proposition 3.** *If each prime divisor of  $k \in \mathbb{N}$  occurs infinitely often in the sequence  $P$ , then the limit mapping  $h_P^k : \Sigma_P \rightarrow \Sigma_P$  is a homeomorphism.*

Let  $n \in \mathbb{N}$  and  $\xi_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ , where  $i$  is the imaginary unit, i.e.,  $i^2 = -1$ . The set  $\sqrt[n]{1}$  is a multiplicative cyclic group  $\{1, \xi_n, \xi_n^2, \dots, \xi_n^{n-1}\}$  generated by  $\xi_n$ . For  $m \in \mathbb{N}$ , we define a homomorphism  $\psi_m : \sqrt[n]{1} \rightarrow \sqrt[m]{1}$  by setting  $\psi_m(\xi_n^j) = \xi_n^{jm}$ , where  $j \in \{1, 2, \dots, n\}$ . Let  $n$  and  $m$  be relatively prime. In this case, one can easily see that  $\psi_m$  is injective. This implies that  $\psi_m$  is an automorphism, i.e., a bijective homomorphism from a group onto itself. Therefore, there exists an automorphism  $\phi_m : \sqrt[m]{1} \rightarrow \sqrt[n]{1}$ , which is the inverse of  $\psi_m$ . In other words, for each  $a \in \sqrt[m]{1}$  we have exactly one element  $b$  of the group  $\sqrt[n]{1}$  whose  $m$ -th power is equal to  $a$ . The value of the automorphism  $\phi_m$  at  $a$  is just the element  $b$ .

**Proposition 4.** *If  $k \in S(P)$ , then the limit mapping  $h_P^k : \Sigma_P \rightarrow \Sigma_P$  is a  $k$ -fold covering mapping.*

*Proof.* Denote by  $m \in \mathbb{N}$  the least number  $l$  such that  $k \neq p_n$  for all  $n \geq l$ .

If  $m = 1$ , i.e.,  $k$  is not a term of the sequence  $P$ , then for each  $p_n \in P$ , the numbers  $p_n$  and  $k$  are relatively prime. Let  $\phi_{p_n}$  be the inverse of the automorphism  $\psi_{p_n} : \sqrt[k]{1} \rightarrow \sqrt[k]{1} : z \mapsto z^{p_n}$ . We have  $(\phi_{p_n}(z))^{p_n} = z$  for every  $z \in \mathbb{S}^1$ . It is easy to see that for each  $k$ -th root of unity  $\xi_k^j$ , where  $j \in \{1, 2, \dots, k\}$ , the element

$$(\xi_k^j, \phi_{p_1}(\xi_k^j), \phi_{p_2} \circ \phi_{p_1}(\xi_k^j), \dots)$$

lies in the set  $(h_P^k)^{-1}(e)$ . Clearly, in this way one obtains  $k$  distinct points of the fiber  $(h_P^k)^{-1}(e)$ . Hence  $\text{card}(h_P^k)^{-1}(e) \geq k$ . On the other hand, by Proposition 1,  $\text{card}(h_P^k)^{-1}(e) \leq k$  and the desired conclusion follows.

If  $m > 1$ , then we consider the sequence  $Q = (p_m, p_{m+1}, \dots)$  which is equivalent to  $P$ . As mentioned in Introduction, there exists a topological isomorphism  $\rho : \Sigma_P \rightarrow \Sigma_Q$  of topological groups (take  $\rho$  to be the shift of the form  $\rho((x_1, x_2, \dots)) = (x_m, x_{m+1}, \dots)$ , where  $(x_1, x_2, \dots) \in \Sigma_P$ ). The inverse of  $\rho$  is denoted by  $\tau$ .

One can readily verify that the following diagram

$$\begin{array}{ccc} \Sigma_P & \xrightarrow{\rho} & \Sigma_Q \\ h_P^k \downarrow & & \downarrow h_Q^k \\ \Sigma_P & \xrightarrow{\rho} & \Sigma_Q \end{array} \quad (3)$$

is commutative, i.e.  $\rho \circ h_P^k = h_Q^k \circ \rho$ . According to the first part of this proof, the limit mapping  $h_Q^k$  is a  $k$ -fold covering mapping.

It follows from the commutativity of the diagram (3) that

$$(h_P^k)^{-1}(e) = (h_Q^k)^{-1}(\tau(e)) = \tau((h_Q^k)^{-1}(e)).$$

But the mapping  $\tau$  is injective and  $\text{card}(h_Q^k)^{-1}(e) = k$ . Thus the equality  $\text{card}(h_P^k)^{-1}(e) = k$  holds. Therefore, by Proposition 1,  $h_P^k$  is a  $k$ -fold covering mapping.  $\square$

As an immediate consequence of the above results we have the following theorem.

**Theorem 1.** *Let  $k \in \mathbb{N}$  be given. The  $P$ -adic solenoid admits a  $k$ -fold covering mapping of the form  $h_P^k$  if and only if  $k$  has no a prime divisor which occurs infinitely often in the sequence  $P$ .*

It is interesting to restate Theorem 1. To do this we recall some facts. We refer to [11] for a basic material on topological groups. The character group of  $\Sigma_P$ , consisting of all continuous homomorphisms from  $\Sigma_P$  into  $\mathbb{S}^1$ , is topologically isomorphic to the discrete group  $F_P$  of  $P$ -adic rationals (all rationals of the form  $m/(p_1 p_2 \dots p_n)$ , where  $m$  is an integer and  $n \in \mathbb{N}$ ) [11, (25.3)]. It follows from the Pontrjagin duality that two solenoids  $\Sigma_P$  and  $\Sigma_Q$  are topologically isomorphic if and only if the additive groups  $F_P$  and  $F_Q$  are isomorphic.

An additive abelian group  $G$  is said to be  $n$ -divisible provided that for each element  $g \in G$  there is an element  $g' \in G$  such that  $ng' = g$ . By number-theoretic considerations one can see that, for a prime number  $p$ , the group  $F_P$  is  $p$ -divisible if and only if  $p$  occurs infinitely often in  $P$ .

**Theorem 1'.** *Let  $k \in \mathbb{N}$  be given. The  $P$ -adic solenoid admits a  $k$ -fold covering mapping of the form  $h_P^k$  if and only if  $k$  has no a prime divisor  $q$  such that the group of  $P$ -adic rationals is  $q$ -divisible.*

*Remark 1.* Let  $k \geq 2$  and let  $G$  be a compact connected abelian group. Suppose that the character group of  $G$  is  $k$ -divisible. Using Theorem 1 of [9], it can be shown that there is no a  $k$ -sheeted covering mapping from a connected Hausdorff topological space onto  $G$  [10].

In what follows a *finite-sheeted connected covering of a solenoid* means a finite-sheeted covering mapping from a connected space onto a solenoid. We conclude this section with a theorem which summarizes the results concerning finite-sheeted connected coverings of solenoids. For each  $k \in \mathbb{N}$  that is not divisible by any of the primes that occur infinitely often in the sequence  $P$  there is just one (up to equivalence)  $k$ -fold connected covering of  $\Sigma_P$ , and these are the only finite-sheeted connected coverings of  $\Sigma_P$  (see [7, Example 2], [8, p. 82, Theorem 3] and [15, Proposition 2.2]). This fact is a corollary of the theory of overlays (We refer to [7], [8] and [12] for this theory). According to the above results we have the following theorem (with the same dichotomy of positive integers  $k$  as in Theorems 1 and 1').

**Theorem 2.** *Let  $P$  be a sequence of prime numbers and  $k \in \mathbb{N}$ . If  $k$  is a multiple of some prime number which occurs infinitely often in  $P$ , then there is no a  $k$ -fold connected covering of the  $P$ -adic solenoid  $\Sigma_P$ . Otherwise, the limit mapping  $h_P^k$  is a  $k$ -fold connected covering and, moreover, each  $k$ -fold connected covering of  $\Sigma_P$  is equivalent to  $h_P^k$ .*

*Remark 2.* The fact that each finite-sheeted connected covering of  $\Sigma_P$  is isomorphic to some limit mapping  $h_P^k$  (and, as a consequence, Theorem 2) can be proved without using the theory of overlays. In order to show this we make use of the approximate construction sketched in [9] (A detailed account of this construction for a finite-sheeted covering mapping from a connected Hausdorff topological space onto an arbitrary compact connected group is contained in [10]). For the sake of completeness, we outline the proof of the above-mentioned fact as follows.

Let  $f : X \rightarrow \Sigma_P$  be a  $k$ -fold connected covering. There exists an inverse sequence  $\{X_n, g_n^{n+1}\}$  and a mapping  $\{g_n : X_n \rightarrow S^1 \mid n \in \mathbb{N}\}$  between  $\{X_n, g_n^{n+1}\}$  and the inverse sequence (1) such that the properties listed below are fulfilled: 1) for each  $n \in \mathbb{N}$  the space  $X_n$  is connected and locally pathwise connected and  $g_n : X_n \rightarrow S^1$  is a  $k$ -fold covering mapping; 2) the  $k$ -fold covering mapping  $f$  is isomorphic to the limit mapping  $g_\infty : X_\infty \rightarrow \Sigma_P$  induced by  $\{g_n : X_n \rightarrow S^1\}$ ; 3) for each  $n \in \mathbb{N}$  there is a point  $x_n \in X_n$  such that  $g_n^{n+1}(x_{n+1}) = x_n$  and  $g_n(x_n) = 1$ . Using the classical covering space theory (see, e.g., [13, Chapter V, Corollary 6.4]), one constructs a sequence  $\{\phi_n : S^1 \rightarrow X_n\}$  of homeomorphisms such that  $\phi_n(1) = x_n$  and  $g_n \circ \phi_n = h_n^k$ , where  $h_n^k$  is the  $k$ -fold covering mapping from (2). In view of uniqueness of liftings [13, Chapter V, Lemma 3.2], it is easy to see that  $\{\phi_n : n \in \mathbb{N}\}$  is a mapping between the inverse sequences (1) and  $\{X_n, g_n^{n+1}\}$ . Since each mapping  $\phi_n : S^1 \rightarrow X_n$  is a homeomorphism, the limit mapping  $\phi_\infty : \Sigma_P \rightarrow X_\infty$  induced by  $\{\phi_n : n \in \mathbb{N}\}$  is a homeomorphism too [6, Proposition 2.5.10]. It is clear that  $g_\infty \circ \phi_\infty = h_P^k$ . In other words, the limit mappings  $g_\infty$  and  $h_P^k$  are isomorphic. Hence the covering mapping  $f : X \rightarrow \Sigma_P$ , being isomorphic to  $g_\infty : X_\infty \rightarrow \Sigma_P$ , is isomorphic to the limit mapping  $h_P^k : \Sigma_P \rightarrow \Sigma_P$  as well.

### 3. PERIODIC POINTS

Let  $X$  be a space and let  $f : X \rightarrow X$  be a mapping. For  $n \in \mathbb{N}$ , the composite  $f \circ f \circ \dots \circ f$  ( $n$  times) is called the  $n$ -th iteration of  $f$  and is denoted by  $f^n$  ( $f_1 = f$ ). A point  $x \in X$  is said to be *periodic* if there exists  $n \in \mathbb{N}$  such that  $f^n(x) = x$ . In this case,  $n$  is called the *period* of  $x$  under  $f$ .

In this section we shall prove three propositions which correspond to the following cases: 1)  $S(P) = \emptyset$ ; 2)  $S(P)$  is an infinite set; 3)  $S(P)$  is a nonempty finite set.

We first remark that for any sequence of prime numbers  $P$  the limit mapping  $h_P^1$  is the identity on the  $P$ -adic solenoid  $\Sigma_P$ . Therefore, the set of periodic points of  $h_P^1$  coincides with the whole space  $\Sigma_P$ .

**Proposition 5.** *Let  $P = (p_1, p_2, \dots)$  be a sequence of prime numbers such that each prime number from  $\mathbb{N}$  occurs infinitely often in  $P$ . Then, for each  $k \geq 2$ , the identity  $e$  of the  $P$ -adic solenoid is the only periodic point of the limit mapping  $h_P^k$ .*

*Proof.* Fix  $k \geq 2$  and suppose that  $z = (z_1, z_2, \dots)$  is a point of  $\Sigma_P$  such that  $(h_P^k)^m(z) = h_P^{km}(z) = z$  for some  $m \in \mathbb{N}$ . Then, for each  $n \in \mathbb{N}$ , we have  $z_n^{km-1} = 1$  with  $km \geq 2$ . If  $km = 2$ , then  $z = e$ . Let  $km \geq 3$ . For given  $n \in \mathbb{N}$ , we choose  $l > n$  such that the product  $p_n p_{n+1} \dots p_{l-1}$  of the terms of  $P$  is a multiple of  $km - 1$ . Then

$$z_n = f_n^l(z_l) = z_l^{p_n p_{n+1} \dots p_{l-1}} = 1. \quad (4)$$

Since this is valid for each  $n \in \mathbb{N}$ , we have  $z = e$ , as required.  $\square$

The next proposition is based on [3, Proposition 40] (cf. [17, Proposition 9]). Note that, in the case of the dyadic solenoid, the set  $S((2, 2, \dots))$  is infinite. Before coming to Proposition 6 we recall Euler's Theorem. It states that for relatively prime positive integers  $a$  and  $m$  the number  $a^{\varphi(m)} - 1$  is divisible by  $m$ . Here,  $\varphi(m)$  is a value of the Euler function at  $m$ . That is,  $\varphi(1) = 1$  and, for  $m > 1$ ,  $\varphi(m) = q_1^{l_1-1} q_2^{l_2-1} \dots q_n^{l_n-1} (q_1 - 1)(q_2 - 1) \dots (q_n - 1)$ , where  $m = q_1^{l_1} q_2^{l_2} \dots q_n^{l_n}$  is the canonical factorization of  $m$ , i.e.,  $q_1, q_2, \dots, q_n \geq 2$  are distinct prime numbers and  $l_1, l_2, \dots, l_n \in \mathbb{N}$ .

**Proposition 6.** *Let  $P = (p_1, p_2, \dots)$  be a sequence of prime numbers such that the set  $S(P)$  is infinite. Then, for each  $k \in \mathbb{N}$ , the set of all periodic points of the limit mapping  $h_P^k$  is dense in  $\Sigma_P$ .*

*Proof.* Given  $k > 1$ , we choose  $q \in S(P)$  such that  $q > k$ . Let  $\pi_n : \Sigma_P \rightarrow \mathbb{S}^1 : (z_1, z_2, \dots) \mapsto z_n$  be the  $n$ -th projection mapping of the inverse sequence (1),  $n \in \mathbb{N}$ .

Take a basic open subset  $U$  of  $\Sigma_P$ . That is,  $U = \pi_l^{-1}(V_l)$  for some  $l \in \mathbb{N}$  and some open subset  $V_l$  of  $\mathbb{S}^1$ . Recall that for any  $n \in \mathbb{N}$  such that  $n \geq l$ , the projection mappings  $\pi_n$  and  $\pi_l$  satisfy the equality  $\pi_l = f_l^n \circ \pi_n$ . Therefore, for each  $n \geq l$ , we have  $U = \pi_l^{-1}(V_l) = \pi_n^{-1}(V_n)$ , where  $V_n = (f_l^n)^{-1}(V_l)$  is an open subset of  $\mathbb{S}^1$ . Since  $q \in S(P)$  we can choose  $n \geq l$  such that  $p_j \neq q$  for every  $j \geq n$ . Fix such  $n$ .

Take  $m \in \mathbb{N}$  such that the following condition is fulfilled:

$$\text{there exists a point } z_n \in V_n \text{ such that } z_n^{q^m} = 1. \quad (5)$$

For each  $j \geq n$  the numbers  $p_j$  and  $q^m$  are relatively prime. Consider the automorphism  $\psi_{p_j} : \sqrt[q^m]{1} \rightarrow \sqrt[q^m]{1} : z \mapsto z^{p_j}$  and its inverse  $\phi_{p_j}$ . Thus we have

$$(\phi_{p_j}(z))^{p_j} = z \text{ for all } z \in \sqrt[q^m]{1}. \quad (6)$$

Let  $\zeta = (z_n^{p_1 p_2 \dots p_{n-1}}, \dots, z_n^{p_{n-1}}, z_n, \phi_{p_n}(z_n), \phi_{p_{n+1}} \circ \phi_{p_n}(z_n), \dots)$ . According to (6), the point  $\zeta$  belongs to  $\Sigma_P$ . Moreover, by (5),

$$\zeta \in U. \quad (7)$$

We claim that  $\zeta$  is a periodic point of  $h_P^k$ . Indeed, since each term of the sequence  $\zeta$  is a  $q^m$ -th root of 1, we get

$$\zeta^{q^m} = e. \quad (8)$$

Since  $k$  and  $q^m$  are relatively prime, by Euler's Theorem,  $k^{q^{m-1}(q-1)} - 1$  is divisible by  $q^m$ . Hence, by (8),

$$\zeta^{k^{q^{m-1}(q-1)} - 1} = e \quad \text{and} \quad (h_P^k)^{q^{m-1}(q-1)}(\zeta) = \zeta^{k^{q^{m-1}(q-1)}} = \zeta.$$

In other words,  $\zeta$  is a point of period  $q^{m-1}(q-1)$  under  $h_P^k$ .

In view of (7) and since  $U$  was chosen as an arbitrary basic open subset of  $\Sigma_P$ , the proof is complete.  $\square$

**Proposition 7.** *Let  $P = (p_1, p_2, \dots)$  be a sequence of prime numbers such that the set  $S(P)$  is nonempty and finite. Let  $k \in \mathbb{N}$  be given. If  $k$  is a multiple of the product of all prime numbers from  $S(P)$ , then the identity  $e$  of the  $P$ -adic solenoid is the only periodic point of the limit mapping  $h_P^k$ . If there exists a prime number from  $S(P)$  which is not a divisor of  $k$ , then the set of all periodic points of the limit mapping  $h_P^k$  is dense in  $\Sigma_P$ .*

*Proof.* Suppose that  $S(P) = \{q_1, \dots, q_t\}$ , where  $t \in \mathbb{N}$ .

First, let  $k$  be a multiple of  $q_1 \cdot \dots \cdot q_t$ , and let  $z = (z_1, z_2, \dots) \in \Sigma_P$  be a periodic point of period  $m \in \mathbb{N}$  under  $h_P^k$ . Thus, for all  $n \in \mathbb{N}$ , we have  $z_n^{k^m - 1} = 1$ , where  $k^m \geq 2$ . If  $k^m = 2$ , then  $z = e$ . If  $k^m \geq 3$ , then we choose  $n \in \mathbb{N}$  such that

$$p_j \notin S(P) \quad \text{for all } j \geq n. \quad (9)$$

Obviously, for each  $q_j \in S(P)$ ,  $j \in \{1, \dots, t\}$ , the number  $k^m - 1$  is not divisible by  $q_j$ . Hence, each prime divisor of  $k^m - 1$  occurs infinitely often in the sequence  $P$ . Choose  $l > n$  such that the number  $p_n p_{n+1} \dots p_{l-1}$  is a multiple of  $k^m - 1$ . Then we have  $z_n = 1$  (see (4)). This implies  $z_l = f_l^n(z_n) = 1$  for every  $l \leq n$ . Since this is true for each  $n$  satisfying (9), we get  $z = e$ .

Second, we assume that  $k > 1$  is not divisible by  $q \in S(P)$ . Then every power of  $q$  and  $k$  are relatively prime. Repeating the proof of Proposition 6 for these  $k$  and  $q$ , one can see that the set of periodic points of  $h_P^k$  is dense in  $\Sigma_P$ .  $\square$

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